Rotational Dynamics

The Parallel Axis Theorem
Right Hand Rule: The real direction of CW and CCW rotation
Dot and Cross Products
Angular Momentum
Conservation of Angular Momentum
Collisions with Fixed Pivots
Rotational Inertia list

- Rod: \( \frac{1}{12}ml^2 \)
- Thin ring: \( mr^2 \)
- Solid disk: \( \frac{1}{2}mr^2 \)
- Hollow sphere: \( \frac{2}{3}mr^2 \)
- Solid sphere: \( \frac{2}{5}mr^2 \)
**Parallel Axis Theorem**

Calculates the new rotational inertia when you move the pivot away from the CM

\[
I_{\text{NEW}} = I_{\text{CM}} + md^2
\]

- Old rotational inertia around center of mass
- Mass of object
- Distance the pivot was shifted
\[ I_{\text{NEW}} = I_{\text{CM}} + md^2 \]

Use the Parallel Axis Theorem to show that the rotational inertia of a rod around its end is \( \frac{1}{3} mL^2 \)

\[ I_{\text{new}} = I_{\text{CM}} + md \]

\[ = \frac{1}{12} mL^2 + m \left( \frac{L}{2} \right)^2 \]

\[ = \frac{1}{12} mL^2 + \frac{1}{4} mL^2 \]

\[ = \frac{1}{12} mL^2 + \frac{3}{12} mL^2 \]

\[ = \frac{4}{12} mL^2 = \frac{1}{3} mL^2 \checkmark \]
The Real Direction of CW and CCW

The Right Hand Rule

$\vec{\omega}$

$\vec{\omega} (+2)$

$\vec{\omega}$
1st Right Hand Rule: Angular Velocity

Stick your right thumb out. Curl the fingers of your right hand with the rotation; your thumb points in the direction of the angular velocity.
1st Right Hand Rule: Angular Velocity
Stick your right thumb out. Curl the fingers of your right hand with the rotation; your thumb points in the direction of the angular velocity.
Two Ways to Multiply Vectors

\[ \vec{A} \times \vec{B} \]

Cross Product

\[ AB \sin \theta \]

When perpendicularity matters

\[ \vec{T} = r \vec{F} \sin \theta \]

\[ \vec{L} = r \times \vec{F} \]

vector

\[ \vec{A} \cdot \vec{B} \]

Dot Product

\[ AB \cos \theta \]

When parallel-ness matters

\[ W = F \vec{s} \cos \theta \]

\[ W = \vec{F} \cdot \vec{s} \]

Scalar
2nd Right Hand Rule: Cross Product

Orient the index finger, middle finger and thumb of your right hand to be at 90 degrees to each other.

- The index finger points with the 1st vector in the product.
- The middle finger points with the 2nd vector in the product.
- Your thumb will point in the direction of the cross product answer.
2nd Right Hand Rule: Cross Product

Move only your wrist (not your fingers) so that your:

- index finger points with the 1st vector in the product.
- middle finger points with the 2nd vector in the product.
- your thumb will point in the direction of the cross product answer.

\[ \vec{\tau} = \vec{r} \times \vec{F} \]
How do you draw a vector coming out of the page or into it?

- Out of page or screen
- Into page or screen

Point

Tail feather
Use the right-hand rule to find the direction of torque.
Find the dot product and the cross product of the vectors below.

\[ \vec{A} = 3 \hat{i} + 4 \hat{j} - 5 \hat{k} \]
\[ \vec{B} = 2 \hat{i} - 6 \hat{j} + 1 \hat{k} \]

\[ \vec{A} \cdot \vec{B} = (3)(2) + (4)(-6) + (-5)(1) \]
\[ = 6 - 24 - 5 \]
\[ = -23 \]
Find the dot product and the cross product of the vectors below.

\[ \vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k} \]

\[ \vec{B} = 2\hat{i} - 6\hat{j} + 1\hat{k} \]

\[ \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -5 \\ 2 & -6 & \end{vmatrix} = \hat{i} \left( (4)(1) - (-5)(-6) \right) - \hat{j} \left( (3)(1) - (-2)(-5) \right) + \hat{k} \left( (3)(-6) - (-2)(-5) \right) \]

\[ = (4-30)\hat{i} - (3+10)\hat{j} + (-18-8)\hat{k} \]

\[ = -26\hat{i} - 13\hat{j} - 26\hat{k} \]
Find the work done by the force over the displacement given.

\[ \vec{F} = 3 \hat{i} + 2 \hat{j} \]
\[ \vec{s} = 6 \hat{i} + 0 \hat{j} \]

\[ W = \vec{F} \cdot \vec{s} = (3)(6) + (2)(0) = 18 + 0 = 18 \text{ J} \]

Scalar
Find the torque for the force and radius vectors below.

\[
\overrightarrow{F} = \left[ 2\hat{i} + 3\hat{j} + 5\hat{k} \right]_m
\]

\[
\overrightarrow{F} = \left[ -1\hat{i} + 2\hat{j} - 3\hat{k} \right]_N
\]

\[
\overrightarrow{r} \times \overrightarrow{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & 3 & 5 \\
-1 & 2 & -3
\end{vmatrix} = \hat{i}(3 \cdot 5 - 2 \cdot 5) - \hat{j}((2 \cdot 5 - (-1) \cdot 5)) + \hat{k}((2 \cdot (-3) - (-1) \cdot 3))
\]

\[
= (-9 - 10)\hat{i} - (-6 + 5)\hat{j} + (4 + 3)\hat{k}
\]

\[
= [-19\hat{i} + 1\hat{j} + 7\hat{k}]_m N
\]
Using \( r \)-perpendicular to Calculate Torque

\[ \tau = rF \sin \theta \]

\[ \tau = r_\perp F \]
Linear vs Angular Momentum

\[ \vec{p} = m \vec{v} \]

\[ \vec{L} = I \vec{\omega} \]

\[ \vec{L} = \vec{r} \times \vec{p} \]
Find the angular momentum of a disk, mass 10 kg, radius 2 meters, rotating clockwise about its CM at 5 rad/s.

\[
\vec{L} = I \vec{\omega}
\]

\[
L = I \omega = \left( \frac{1}{2} m r^2 \right) \omega
\]

\[
= \frac{1}{2} (10)(2)^2 (5)
\]

\[
= -100 \text{ kg m}^2 \text{s}^{-1} \hat{k}
\]
Does a LINEARLY moving object have ANGULAR momentum?

\[ \vec{L} = \vec{r} \times \vec{p} \]

\[ L = r \, m \, v \, \sin \theta \]
The ball moves with the constant velocity shown. Find its angular momentum. Its current position: (-6, 4) m.

\[
\hat{\mathbf{L}} = \mathbf{r} \times \mathbf{p} \\
\mathbf{L} = r_\perp \mathbf{p} \\
= (4\text{m})(2.5\text{m/s})(3\text{m}^3) \\
= -24 \text{ kgm}^2/\text{s} \hat{k}
\]
Law of Conservation of Angular Momentum

If there are no outside torques, angular momentum is conserved.

\[ \vec{L}_i = \vec{L}_f \]
Find the angular velocity after the totally inelastic collision. The bar has a fixed pivot at the top.

(subquestion: why can't you use linear momentum?)

\[ \mathbf{L}_i = \mathbf{L}_f \]

\[ r_\perp \rho = I \omega \]

\[(1m)(2.5)(3.75) = (I_{\text{bar}} + I_{\text{ball}}) \omega\]

\[ 6 \text{ kgm}^2 = \left( \frac{1}{3} (0.63)(1)^2 + (2.5)(1)^2 \right) \omega \]

\[ 6 \text{ kgm}^2 = \left( 0.2 \text{ kgm}^2 + 2 \text{ kgm}^2 \right) \omega \]

\[ 6 \text{ kgm}^2 = (2.2 \text{ kgm}^2) \omega \]

\[ 2.72 \text{ rad} \frac{1}{s} = \omega \]
**Force and Momentum; Torque and Angular Momentum**

\[ \vec{F} = \frac{d\vec{p}}{dt} \quad \text{and} \quad \vec{\tau} = \frac{d\vec{L}}{dt} \]

\[ \frac{d}{dt} (m\vec{v}) \]

\[ = m \frac{d\vec{v}}{dt} \]

\[ = ma \]
What is the Torque at t = 2 seconds?

\[ \vec{\tau} = (5t^2) \hat{i} + (2t+1) \hat{j} + (4) \hat{k} \]

\[ \vec{\tau} = \frac{d\vec{\omega}}{dt} = (10t) \hat{i} + (2) \hat{j} \]

\[ \vec{\tau} (2) = (20 \hat{i} + 2 \hat{j}) \text{ mN} \]
**Torque and Rolling**

The yo-yo is unwinding and falling while the top of the string is being held in place. The mass of the yo-yo is 0.02 kg. What is the acceleration of the yo-yo? Assume that the string unwinds from the edge and that the yo-yo is approximately a disk.

\[ \sum F = ma \]

\[ mg - T = ma \]

\[ 0.02 \times 10 - T = 0.02a \]

\[ 0.2 - T = 0.02a \]

\[ 0.2 - 0.01a = 0.02a \]

\[ 0.2 = 0.03a \]

\[ \frac{6.67n^2}{3^2} = a \]

\[ T = \frac{1}{2} ma \]

\[ T = 0.01a \]
**Torque and Rolling**

The yo-yo is unwinding and falling while the top of the string is being held in place. The mass of the yo-yo is 0.02 kg. What is the acceleration of the yo-yo? Assume that the string unwinds from the edge and that the yo-yo is approximately a disk.

\[ V = rw \quad a = r \alpha \]

\[ \sum T = I \alpha \]

\[ rmg = \left(\frac{3}{2}mr^2\right) \left(\frac{a}{r}\right) \]

\[ I_{nw} = I_{cm} + m\ell^2 \]

\[ \begin{align*}
I_{nw} &= \frac{1}{2}mr^2 + mr^2 \\
&= \frac{3}{2}mr^2 \\
mg &= \frac{3}{2}ma \\
\frac{2}{3}g &= a \\
T &= 0.01a
\end{align*} \]