Rotational Motion

World of Translation		World of Rotation
[m] x, y		θ (radians)
(m/s) v	$v = r\omega$	ω (rad/s)
(m/s²) a	$a = r\alpha$	α (rad/s²)

equations of kinematics

constant (angular) acceleration only!

$$\Delta x = v_0 t + \tfrac{1}{2} a t^2 \qquad \Delta\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\Delta x = \tfrac{1}{2}(v + v_0)t \qquad \Delta\theta = \tfrac{1}{2}(\omega + \omega_0)t$$

$$v = v_0 + at \qquad \omega = \omega_0 + \alpha t$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

Rotational Motion - Calculus

$$v = \frac{dx}{dt}$$

$$\omega = \frac{d\theta}{dt}$$

$$x = \int v\,dt$$

$$\theta = \int \omega\,dt$$

$$a = \frac{dv}{dt}$$

$$\alpha = \frac{d\omega}{dt}$$

$$v = \int a\,dt$$

$$\omega = \int \alpha\,dt$$

$\Delta\theta$ ω α

2π radians $1\ \text{rad}/\text{s}$ $2\ \text{rad}/\text{s}^2$

4 rotations $25\ \frac{\text{revolutions}}{\text{min}}$

ex: converting rotations

θ

100 rotations

$$100 \text{ rotations} \times \frac{2\pi \text{ rad}}{1 \text{ rotation}} = 200\pi \text{ rad}$$

$$\approx 628 \text{ rad}$$

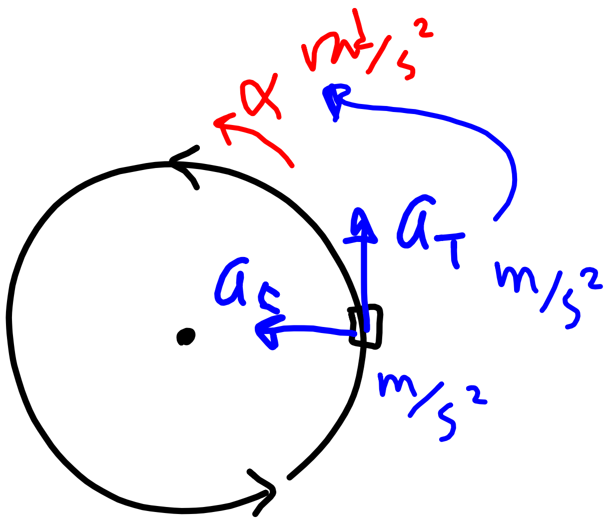
ω

20 rpm

$$20 \frac{\text{rot}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rot}} \times \frac{1 \text{ min}}{60 \text{ sec}} = \frac{2}{3}\pi \text{ rad/s}$$

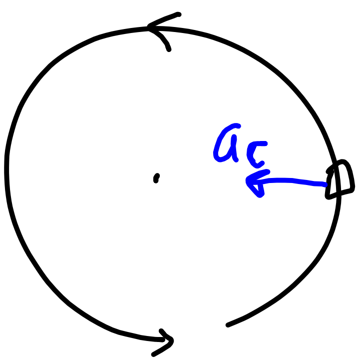
$$\approx 2 \text{ rad/s}$$

Angular vs Linear Accelerations



Uniform Circular Motion - going around at a constant rate

(no α angular acceleration)



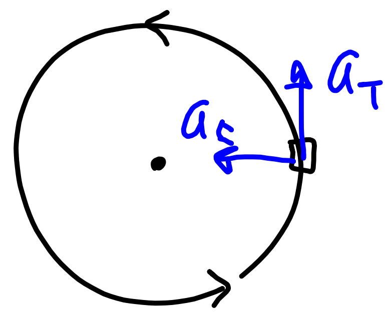
centripetal (radial) accel only

$$a_c = \frac{v^2}{r} = r\omega^2$$

$$a_T = 0$$

Circular Motion - speeding up or slowing down

(yes α angular acceleration)

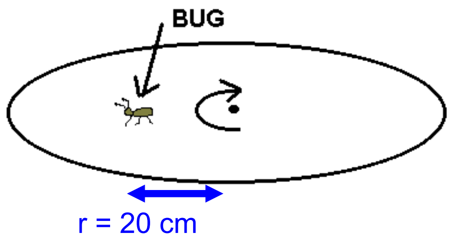


centripetal (radial) accel and tangential accel

$$a_c = \frac{v^2}{r}$$

$$a_T = r\alpha$$

Rotational Kinematics Problem



A bug is on a turntable. The turntable accelerates uniformly from rest, turning through 5 rotations in 2 seconds.

(a) What angle did the turntable turn through?
(b) What was the final angular velocity of the bug?
(c) What was the bug's angular acceleration?
(d) What was the bug's tangential (linear) acceleration?
(e) What is the bug's radial acceleration at t = 2 s?

a) $5\,\text{rot} \times \frac{2\pi\,\text{rad}}{1\,\text{rot}} = \boxed{10\pi\,\text{rad}}$

b) $\omega_0 = 0$ ✓
$t = 2\,s$ ✓
$\Delta\theta = 10\pi\,\text{rad}$ ✓
$\omega = ?$

$\Delta\theta = \frac{1}{2}(\omega + \omega_0)t$
$10\pi = \frac{1}{2}(\omega + 0)(2)$
$\boxed{10\pi\,\text{rad/s} = \omega}$

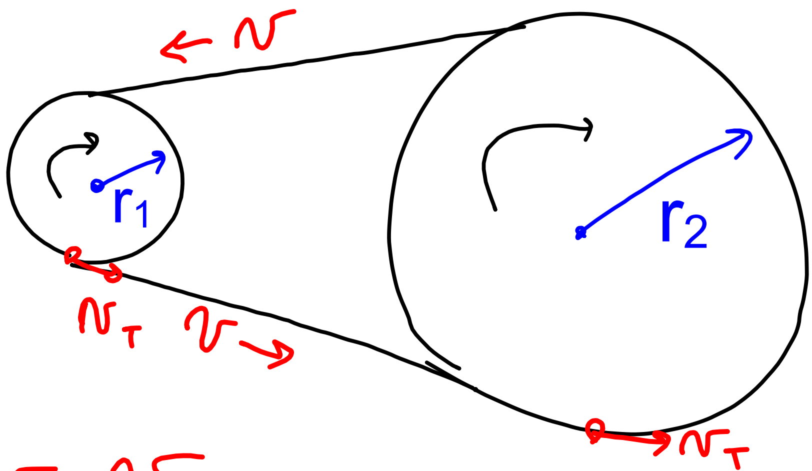
c) $\alpha = ?$

$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$10\pi = (0)(2) + \frac{1}{2}\alpha(2)^2$
$10\pi = 2\alpha$
$5\pi\,\text{rad/s}^2 = \alpha$

d) $a_t = r\alpha$
$= (0.2)(5\pi) = 6.28\,\text{m/s}^2$

e) $a_c = \frac{v^2}{r} = r\omega^2 = (0.2)(10\pi)^2$
$= 197\,\text{m/s}^2$

coupled wheels & gears



$$v_1 = v_2$$

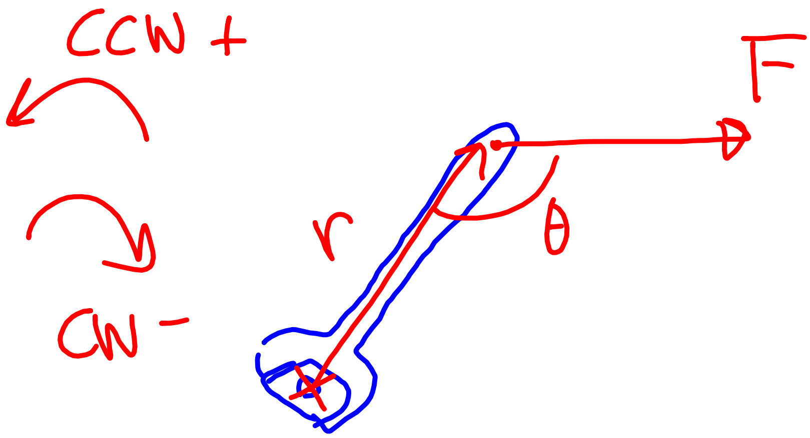
$$r_1 \omega_1 = r_2 \omega_2$$

$$r_1 \omega_1 = r_2 \omega_2$$

$$a_1 = a_2$$

$$r_1 \alpha_1 = r_2 \alpha_2$$

About Torque



$$\tau = rF\sin\theta$$

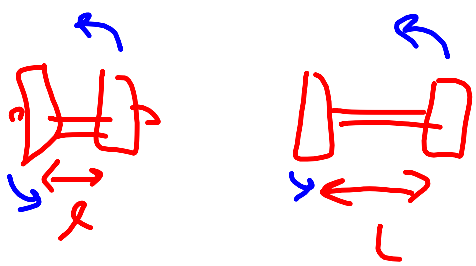
mN

The importance of radius in rotation

$$v = r\omega$$

$$a = r\alpha$$

$$\tau = rF\sin\theta$$



Rotational Inertia
(small objects far from pivot)

$$I = \sum mr^2$$

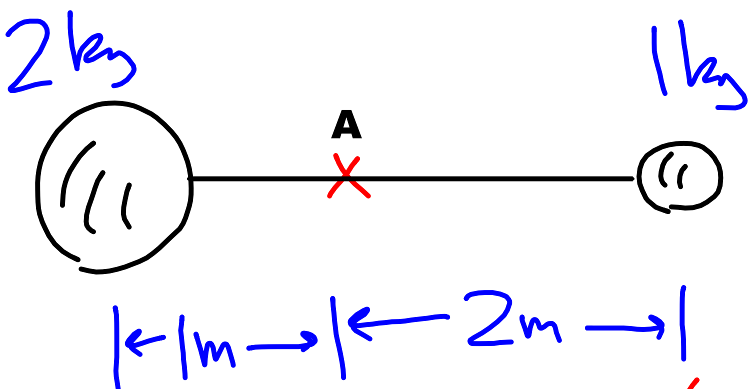
$kg\,m^2$

old terminology: "moment of inertia"

small objects, far from pivot example

Find the rotational inertial of the two spheres around point A.

$$I = \sum mr^2$$



$$I_{total} = (2k)(1m)^2 + (1kg)(2m)^2$$

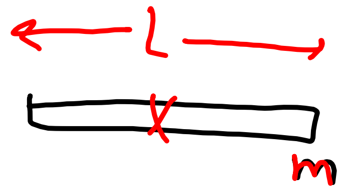
$$= 2kgm^2 + 4kgm^2$$

$$= 6kgm^2$$

Rotational Inertia (extended objects)

$$I = \int r^2 \, dm$$

rod around center



$$I = \frac{1}{12} m L^2$$

rod around end

$$\frac{1}{3} m L^2$$

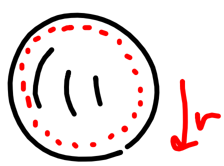
ring

$$m r^2$$

disk

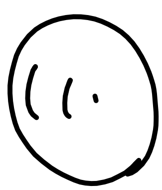
$$\frac{1}{2} m r^2$$

hollow sphere



$$\frac{2}{3} m r^2$$

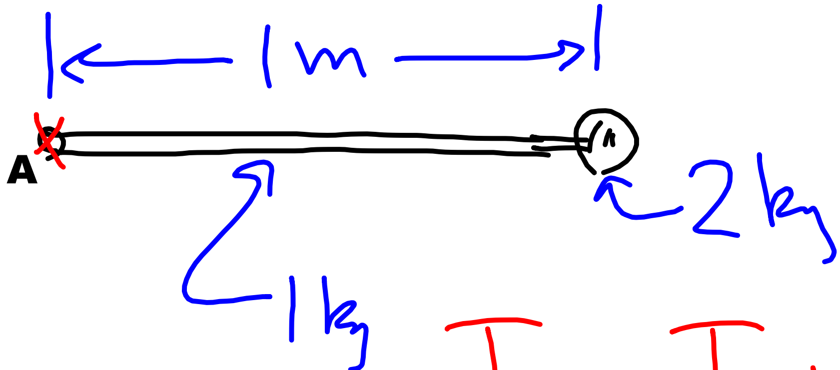
solid sphere



$$\frac{2}{5} m r^2$$

compound objects

What is the rotational inertia of the rod with the ball at the end, pivoting around A? (The ball is small compared to the length of the rod.)



$$I_{total} = I_{rod} + I_{ball}$$

$$= \frac{1}{3} m_{rod} L_{rod}^2 + m_{ball} r_{ball}^2$$

$$= \frac{1}{3}(1)(1)^2 + (2)(1)^2$$

$$= 0.33\ kgm^2 + 2\ kgm^2$$

$$= 2.33\ kgm^2$$

Newton's 2nd Law for Rotation

$$\sum F = ma$$

$$(kg)\left(\frac{m}{s^2}\right)$$

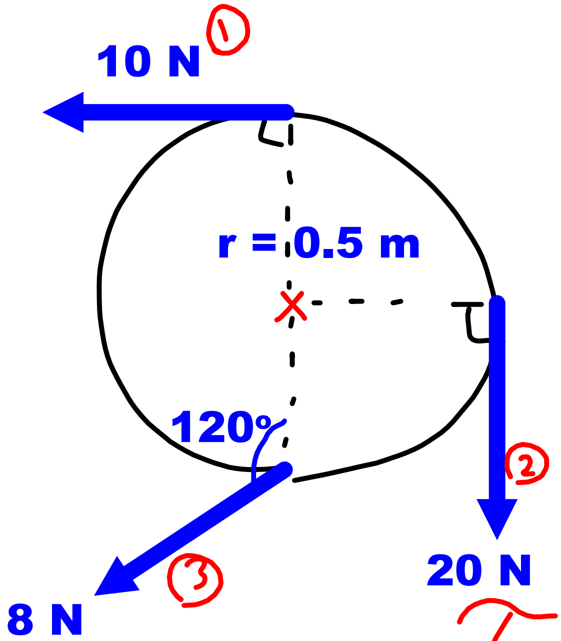
$$\sum \tau = I\alpha$$

$$mN = (kg\,m^2)\,\frac{rad}{s^2}$$

$$= \left(kg\,m \cdot \frac{rad}{s^2}\right)m$$

$$= mN$$

ex: Torque and angular acceleration



a) Find the net Torque

b) If the rotational inertia of the wheel is 5 kg m^2, find the angular acceleration

$$\tau = rF\sin\theta$$

$$\tau_1 = {}^{+}(0.5)(10N)\sin 90 = +5\,mN$$

$$\tau_2 = {}^{-}(0.5)(20N)\sin 90 = -10\,mN$$

$$\tau_3 = {}^{-}(0.5)(8N)\sin 120^\circ = -6.92\,mN$$

$$\boxed{\sum \tau = -11.92\,mN}$$

$$\sum \tau = I\alpha$$

$$-11.92\,mN = (5\,kgm^2)\,\alpha$$

$$\boxed{-1.39\,\frac{rad}{s^2} = \alpha}$$

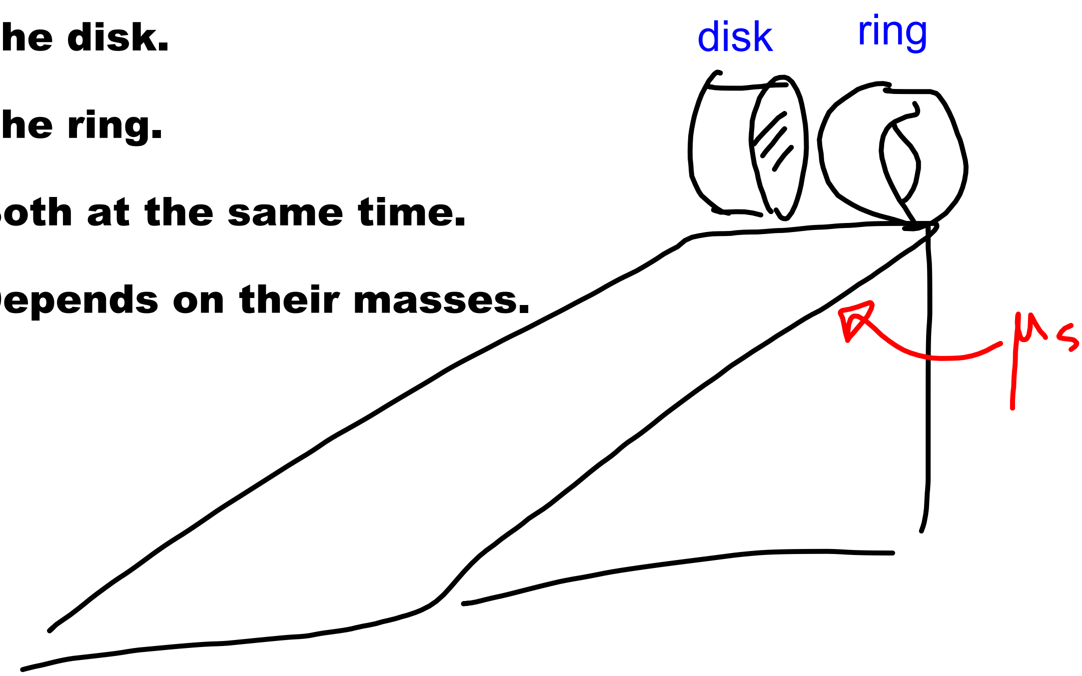
When released from rest, which one will reach the bottom of the ramp first?

a) The disk.

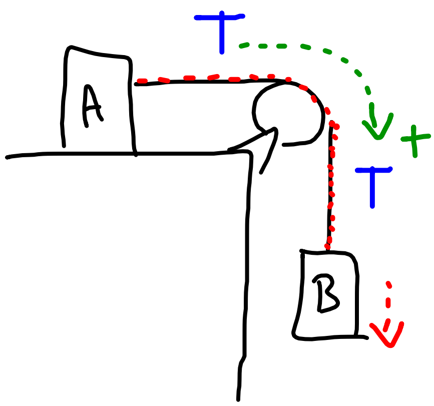
b) The ring.

c) Both at the same time.

d) Depends on their masses.



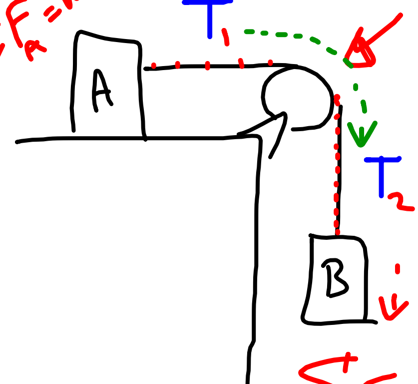
If the pulley has negligible mass...



$a = r\alpha$

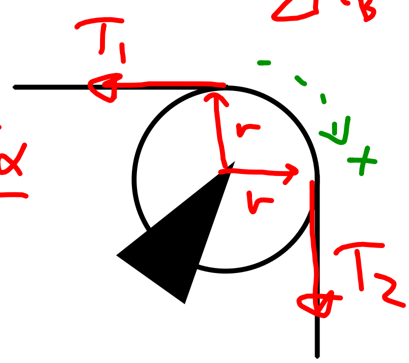
If not...

$\sum F_A = ma$

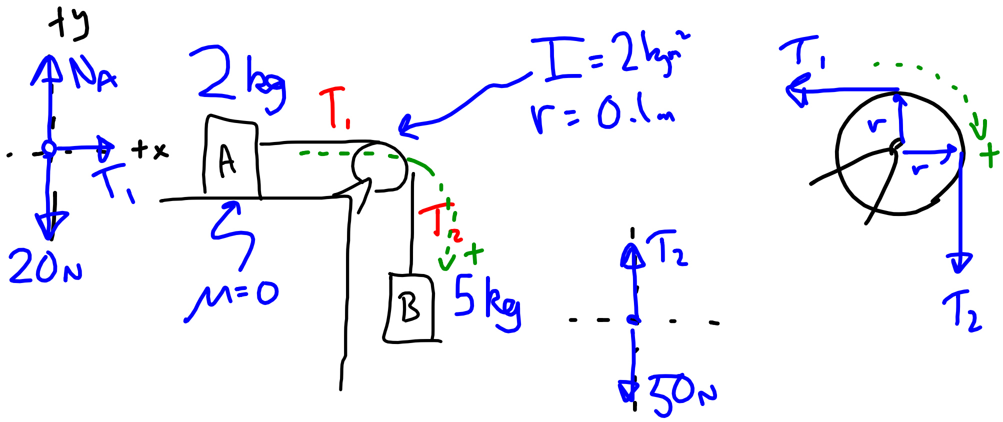


$\sum F_B = ma$

$\sum \tau = I\alpha$



ex: pulley with mass



Find the acceleration of the system.

$\sum F_A = ma$ $\quad \sum F_B = ma$ $\quad \sum \tau = I\alpha$

$T_1 = 2a$ $\quad 50 - T_2 = 5a$

$50 - 5a = T_2$ $\quad rT_2 - rT_1 = I\alpha$

$0.1T_1 + 0.1T_2 = 2\alpha$

$\alpha = \frac{a}{r}$

$0.1T_1 + 0.1T_2 = 2\frac{a}{(0.1)}$

$0.1(T_1 + T_2) = 20a$

$(0.1)(2a + 50 - 5a) = 20a$

$0.2a + 5 - 0.5a = 20a$

$5 - 0.3a = 20a$

$5 = 20.3a$

$0.25\,\frac{m}{s^2} = a$

Translational vs Rotational Kinetic Energy

$K = \frac{1}{2} m v^2$

translational KE

$kg \left(\frac{m}{s}\right)^2$

$kg \frac{m^2}{s^2}$

J

$K = \frac{1}{2} I \omega^2$

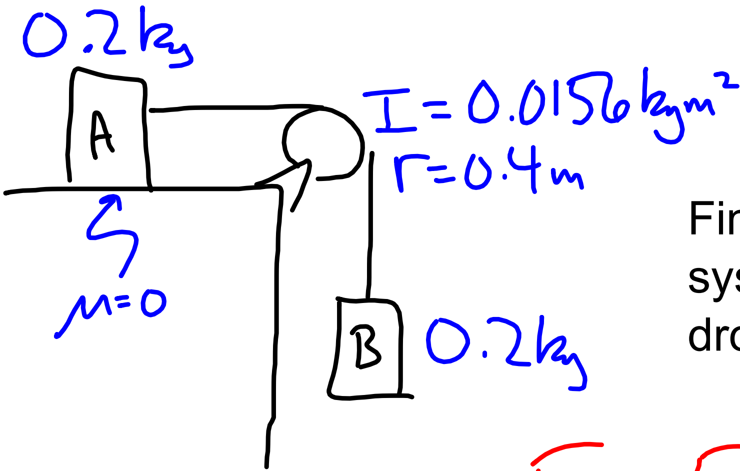
rotational KE

$kg\, m^2 \left(\frac{rad}{s}\right)^2$

$kg \frac{m^2}{s^2}$

J

ex: rotational kinetic energy



Find the speed of the system when B has dropped 1 meter.

$$E_i = E_f$$

$$U_{g_B} = K_B + K_A + K_{pulley}$$

$$m_B g h_B = \frac{1}{2} m_B v^2 + \frac{1}{2} m_A v^2 + \frac{1}{2} I \omega^2$$

$$(0.2)(10)(1) = \frac{1}{2}(0.2)v^2 + \frac{1}{2}(0.2)v^2 + \frac{1}{2}(0.056)\omega^2$$

$$v = r\omega$$

$$2 = 0.1v^2 + 0.1v^2 + \frac{1}{2}(0.0156)\frac{v^2}{(0.4)^2}$$

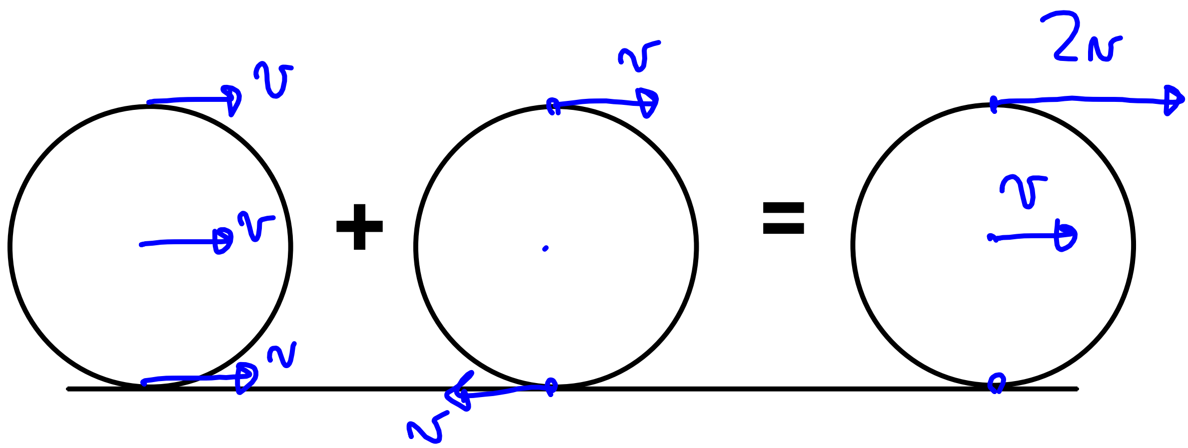
$$2 = 0.2v^2 + 0.04875v^2$$

$$2 = 0.24875v^2$$

$$8.04 = v^2$$

$$\boxed{2.83 \tfrac{m}{s} = v}$$

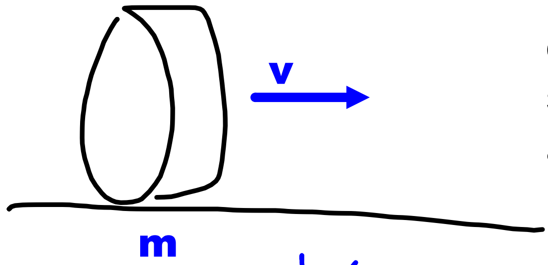
Rolling = Translation + Rotation



Rolling without slipping:

$$\boxed{v = r\omega}$$

ex: rolling kinetic energy



Find the total Kinetic Energy

of the disk as it rolls without slipping in terms of the mass m and the speed of the disk v.

$$K_{tot} = K_T + K_r$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} \left(\frac{1}{2} m r^2\right) \omega^2$$

$v = r\omega$

$$= \frac{1}{2} m v^2 + \frac{1}{4} m (r^2 \omega^2)$$

$$= \frac{1}{2} m v^2 + \frac{1}{4} m v^2$$

$$= \frac{3}{4} m v^2$$

$$\% K_r = \frac{K_r}{K_{tot}} = \frac{\frac{1}{4} m v^2}{\frac{3}{4} m v^2} = \frac{1}{3}$$

33%