PROOF OF THE WORK-ENERGY THEOREM

From Newton's 2nd Law, we have:

$$F = ma$$

Since $a = dv / dt$:

$$F = m\frac{dv}{dt}$$

According to the rules of algebra, we can multiply both sides by the same thing without changing the equation:

$$Fv = mv\frac{dv}{dt}$$

Since $v = dx / dt$:

$$F\frac{dx}{dt} = mv\frac{dv}{dt}$$

Canceling dt from both sides:

$$Fdx = mvdv$$

Integrating both sides:

$$\int F\,dx = \int mv\,dv$$

$\int F\,dx$ is the definition of work:

$$W = \int mv\,dv$$

Assuming m is a constant, and putting in the limits of integration:

$$W = m\int_{vi}^{vf} v\,dv$$

$$W = m\left[\frac{v^2}{2}\right]_{vi}^{vf}$$

$$W = m\left(\frac{v_f^2}{2} - \frac{v_i^2}{2}\right)$$

$$W = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

$$W = \Delta K$$